$$\left|f(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{1}{cx}(c-x)\right|$$

$$= \frac{1}{cx}|c-x|$$

In particular, if $|x-c| < \frac{1}{2}c$

$$\Rightarrow -\frac{1}{2}c < (x-c) < \frac{1}{2}c$$

$$\Rightarrow \frac{1}{2}c < x < \frac{3}{2}c$$

So $\quad 0 < \frac{1}{cx} < \frac{2}{c^2}$ for $|x-c| < \frac{1}{2}c$

Therefore $\left|f(x) - \frac{1}{c}\right| \le \frac{2}{c^2}|x-c| \quad -(1)$

Now $\frac{2}{c^2}|x-c| < \epsilon \Rightarrow |x-c| < \frac{1}{2}c^2\epsilon$

If we choose $\delta(\epsilon) = \inf\{\frac{1}{2}c, \frac{1}{2}c^2\epsilon\}$

then if $0 < |x-c| < \delta(\epsilon)$, it will follow first that $|x-c| < \frac{1}{2}c$ so that (1) is valid

and from (1), $\left|f(x) - \frac{1}{c}\right| < \epsilon$

$$\therefore \lim_{x\to c} f(x) = \frac{1}{c}.$$

(b) Let $f(x) = x^2, \forall x \in R$

Now $|f(x) - c^2| = |x^2 - c^2|$

$$= |(x-c)|\,|(x+c)| \quad -(1)$$

If $|x-c| < 1$, then $|x| \le |c| + 1$

So that $|x+c| \le |x| + |c| \le 2|c| + 1$

From (1), $|f(x) - c^2| \leq (2|c|+1)|x-c| < \epsilon$ —(II)

We take $|x-c| < \dfrac{\epsilon}{(2|c|+1)}$ and

choose $\delta(\epsilon) = \inf\left\{1, \dfrac{\epsilon}{(2|c|+1)}\right\}$

If $0 < |x-c| < \delta$, then from (II)

$|f(x) - c^2| < \epsilon$

$\Rightarrow \lim\limits_{x \to c} f(x) = c^2$.

(c) Let $f(x) = \dfrac{x^3-4}{x^2+1}$, $\forall\ x \in R$

Now $\left|f(x) - \dfrac{4}{5}\right| = \left|\dfrac{x^3-4}{x^2+1} - \dfrac{4}{5}\right|$

$= \dfrac{|5x^2+6x+12|}{5(x^2+1)}|x-2|$ —(I)

Take $|x-2| < 1 \Rightarrow -1 < x-2 < 1$

$\Rightarrow 1 < x < 3$

We have $5x^2+6x+12 \leq 5(3)^2+6.3+12 = 75$

and $5(x^2+1) \geqslant 5(1^2+1) = 10$

$\therefore \dfrac{1}{5(x^2+1)} \leq \dfrac{1}{10}$

From (I) $\left|f(x) - \dfrac{4}{5}\right| \leq \dfrac{75}{10}|x-2| = \dfrac{15}{2}(|x-2|) < \epsilon$ —(II)

Now $\dfrac{15}{2}|x-2| < \epsilon \Rightarrow |x-2| < \dfrac{2}{15}\epsilon$

If $\delta(\epsilon) = \inf\{1, \frac{2}{15}\epsilon\}$, then

$0 < |x-c| < \delta$, ~~it~~

from II $\left|f(x) - \frac{4}{5}\right| < \epsilon$

$\therefore \lim_{x \to 2} f(x) = \frac{4}{5}$.

1.2 One sided limits

Definition 1. A real valued function f is said to tend to a limit l as $x \to c$ from left if for any $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(x) - l| < \epsilon, \text{ when } c-\delta < x < c$$

We write $\lim_{x \to c-0} f(x) = l$ or $f(c-0) = l$ or $\lim_{h \to 0} f(c-h) = l$

l is called left hand limit (LHL) of f at $x=c$

Definition 2. A real valued function f is said to tend to a limit l as $x \to c$ from right if for $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(x) - l| < \epsilon, \text{ when } c < x < c+\delta$$

and ~~write~~ we write $\lim_{x \to c+0} f(x) = l$

or $f(c+0) = l$. or $\lim_{h \to 0} f(c+h) = l$

l is called right hand limit (RHL) of f at $x=c$

Note (1) LHL and RHL are called one sided limits

1.3 Limit at infinity

(1) A real valued function f is said to tend to a limit l as $x \to \infty$ (or $\lim_{x\to\infty} f(x) = l$) if for every $\epsilon > 0$, $\exists$ k $k > 0$, such that

$$|f(x) - l| < \epsilon, \text{ when } x > k$$

2) A real valued function f is said to tend to $+\infty$ as $x \to \infty$ (or $\lim_{x\to\infty} f(x) = +\infty$) if for each $G > 0$ (very large), $\exists$ $k > 0$ s.t.

$$f(x) > G, \text{ when } \cancel{x > k}, x > k.$$

(3) A function f is said to tend to $+\infty$ as $x \to c$ if for every $G > 0$, $\exists$ $\delta > 0$ s.t.

$$f(x) > G, \text{ when } |x - c| < \delta$$

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sequence $\{f(x_n)\}$ exists and equal to l for any sequence $\{x_n\}$, $x_n \neq c$ for any n, converge c

~~i.e.~~ $\cancel{|f(x_n) - l| <}$

~~i.e. for $\epsilon > 0$, $\exists$ $\delta > 0$~~

(2) Divergence Criteria

(a) A number l is not said to limit of function $f(x)$ at $x = c$ if there exists a sequence $\{x_n\}$ with $x_n \neq c$, $\forall n \in N$ such that the sequence $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converge to l.

(b) The function $f(x)$ does not have a limit at $x = c$ if $\exists$ a sequence $\{x_n\}$ with $x_n \neq c$, $\forall n \in N$ such that sequence $\{x_n\}$ converges to c but the sequence $\{f(x_n)\}$ does not converge ~~to~~ in R.

Example Show that $\lim_{x \to 0} \frac{1}{x} \sin\frac{1}{x}$ does not exist.

Solut Let $f(x) = \frac{1}{x} \sin\frac{1}{x}$

The function $f(x)$ is defined for every ~~real~~ x except $x = 0$.

Now let $x_n = \frac{2}{\pi(4n+1)}$, $\forall n \in N$

So $f(x_n) = \frac{\pi(4n+1)}{2} . \sin\left(\frac{\pi}{2} + 2n\pi\right)$

$$= \frac{\pi(4n+1)}{2} . \frac{\pi}{2} \longrightarrow \infty \text{ as } n \to \infty$$

$\Rightarrow \lim_{n \to \infty} f(x_n) = \infty$, when $\{x_n\}$ converges to 0

Again by taking $x_n = \frac{1}{n\pi}$, then

$$f(x_n) = n\pi \sin n\pi = n\pi \cdot 0 = 0, \forall n \in N$$

$$\therefore \lim_{n\to\infty} f(x_n) = 0 \neq \infty, \text{ when } \{x_n\} \text{ converges to zero}$$

Hence $\lim_{x\to 0} f(x)$ does not exist

Q.1. ~~Evaluate~~ Find left hand and right hand limit of the following functions at $x=0$. Also find limits

(a) $\lim_{x\to 0} \frac{x e^{1/x}}{1+e^{1/x}}$ (b) $\lim_{x\to 0} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

(c) $\lim_{x\to 0} \frac{1}{1+e^{-1/x}}$ (d) $\lim_{x\to 0} \frac{1}{1+e^{1/x}}$

Q.2. Evaluate

(a) $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (b) $\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x}$

(c) $\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}}$ (d) $\lim_{x\to 0-} \frac{|x|}{x}$

Q 3. Evaluate

(a) $\lim_{x\to\infty} \frac{x+2}{\sqrt{x}}, x>0$ (b) $\lim_{x\to\infty} \frac{\sqrt{x} - x}{\sqrt{x} + x}, x>0$

(c) $\lim_{x\to\infty} \frac{\sqrt{x} - 5}{\sqrt{x}+3}, x>0$ (d) $\lim_{x\to 0} \frac{1}{x}$.

Q 4. Show that the following limit do not exist

(a) $\lim_{x\to 0} sgn(x)$, where $sgn(x) = \begin{cases} +1, & \text{for } x>0 \\ 0 & \text{for } x=0 \\ -1 & \text{for } x<0 \end{cases}$

(b) $\lim \sin \frac{1}{x}$