Example 4.2 Find $Lf'(0)$ and $Rf'(0)$ for the function

$$f(x) = \begin{cases} x\tan^{-1}\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Solution:

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-h\tan^{-1}\left(-\frac{1}{h}\right) - 0}{-h}$$

$$= \lim_{h\to 0} \tan^{-1}\left(-\frac{1}{h}\right) = -\lim_{h\to 0} \tan^{-1}\left(\frac{1}{h}\right)$$

$$= -\frac{\pi}{2}$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h\tan^{-1}\frac{1}{h} - 0}{h} = \cancel{\tan^{-1}} \cancel{\tan^{-1}\frac{1}{h}}$$

$$= \lim_{h\to 0} \tan^{-1}\left(\frac{1}{h}\right) = \frac{\pi}{2}$$

Example 4.3 Find the derivative of f at $x=0$, where

$$f(x) = x^2|x|$$

Solution: Here $f(x) = x^2|x| = \begin{cases} -x^3, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ x^3, & \text{if } x > 0 \end{cases}$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-(-h)^3 - 0}{-h} = \lim_{h\to 0} -h^2 = 0$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3 - 0}{h} = \lim_{h\to 0} h^2 = 0$$

$$\Rightarrow Lf'(0) = Rf'(0) = 0$$

Hence $f'(0) = 0$

Example 4.4) Show that A function f defined as

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x=0$ and derivative of f is not continuous at $x=0$

Proof: $$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right) - 0}{-h}$$

$$= \lim_{h\to 0} -h \sin\left(-\frac{1}{h}\right)$$

$$= \lim_{h\to 0} h \sin\frac{1}{h} \quad [\because \sin(-x) = -\sin x]$$

$$= 0$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h \sin\frac{1}{h}$$

$$= 0$$

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$\Rightarrow$ $Lf'(0) = Rf'(0)$

$\Rightarrow$ f is differentiable at $x=0$ and $f'(0)=0$

From elementary calculus, we know that

$$f'(x) = 2x \sin\frac{1}{x} - x^2\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right) \text{ for } x \neq 0$$

$$g(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}, \text{ where } g(x) = f'(x)$$

At $x=0$

$$LHL = g(0-0) = \lim_{h\to 0} g(0-h)$$

$$= \lim_{h\to 0}\left[2(-h)\sin\left(-\frac{1}{h}\right) - \cos\left(-\frac{1}{h}\right)\right]$$

$$= \lim_{h\to 0} 2h\sin\frac{1}{h} - \lim_{h\to 0}\cos\frac{1}{h}$$

$$= 0 - \text{a finite value lying b/w } -1 \,\&\, 1$$

$\Rightarrow$ LHL does not exist at $x=0$

Similarly RHL does not exist at $x=0$

Hence $\lim_{x\to 0} g(x)$ does not exist at $x=0$

$\Rightarrow$ $\lim_{x\to 0} f'(x)$ does not exist at $x=0$

$\Rightarrow$ $f'(x)$ is not continuous at $x=0$

Example 4.5. Discuss the derivability of the function

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ x, & x > 1 \end{cases} \quad \text{at } x=1$$

Solution: $$Lf'(1) = \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{1-1}{-h} = 0 \quad [\because f(1)=1]$$

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)-1}{h} = \lim_{h\to 0} \frac{h}{h}$$

$$= 1$$

$$\Rightarrow \quad Lf'(1) \neq Rf'(1)$$

Hence f is not differentiable at $x=1$.

Example 4.6 Discuss the derivability of the function

$$f(x) = \frac{x(e^{\frac{1}{x}} - e^{-\frac{1}{x}})}{(e^{1/x} + e^{-1/x})} \text{ for } x \neq 0 \text{ and } f(0)=0$$

at $x=0$

Solution: Here $f(0)=0$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{f(-h)-0}{-h}$$

$$= \lim_{h\to 0} -\frac{1}{h} \cdot f(-h)$$

$$= \lim_{h\to 0} -\frac{1}{h} \cdot \frac{(-h)(e^{-1/h} - e^{1/h})}{(e^{-1/h} + e^{+1/h})}$$

$$= \lim_{h\to 0} \frac{e^{1/h}(e^{-2/h}-1)}{e^{1/h}(e^{-2/h}+1)} = \lim_{h\to 0} \frac{e^{-2/h}-1}{e^{-2/h}+1}$$

$$= \frac{e^{-2/0} - 1}{e^{-2/0} + 1} = \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = -1$$

$$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{h(e^{1/h} - e^{-1/h})}{(e^{1/h} + e^{-1/h})}$$

$$= \lim_{h \to 0} \frac{\cancel{e^{1/h}}(1 - e^{-2/h})}{\cancel{e^{1/h}}(1 + e^{-2/h})} = \frac{1 - e^{-\infty}}{1 + e^{-\infty}}$$

$$= 1$$

$$\Rightarrow L f'(0) \neq R f'(0)$$

Hence f is derivable at $x = 0$

Example 4.7 If $f(x) = \begin{cases} (x-a) \sin \frac{1}{x-a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$

Then show that $f(x)$ is continuous but not derivable at $x = a$

Solution : Here $f(a) = 0$

$$L f'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

$$= \lim_{h \to 0} \frac{(a-h-a) \sin \frac{1}{a-h-a} - 0}{-h}$$

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$$= \lim_{t_1 \to 0} \frac{t_1 \sin \frac{1}{t_1}}{-t_1}$$

$$= - \lim_{t_1 \to 0} \sin \frac{1}{t_1}$$

$=$ − a finite value oscillating b/w -1 & 1

$\Rightarrow$ $Lf'(a)$ does not exist at $x = a$

Similarly $Rf'(a)$ does not exist at $x = a$

Hence f is not differentiable at $x = a$.