Theorem 4.3 [Derivative of the inverse function]

Let $f : I \to R$, where I be an interval is one-one function and differentiable at $c \in I$ with $f'(c) \neq 0$ then the inverse of the function f is differentiable at $f(c)$ and its derivative at $f(c)$ is $\frac{1}{f'(c)}$

Proof. If g = inverse of f, then

$g : R \to I$ such that $f(x) = y \Leftrightarrow g(y) = x$

Let $y = f(x)$ and $y_0 = f(c)$

Since f is diff. at c, therefore

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$\Rightarrow \quad f(x) - f(c) = (x-c)[f'(c) + \alpha(x)] \quad \cdots (I)$$

where $\alpha(x) \to 0$ as $x \to c$

Now $\quad g(y) - g(y_0) = x - c$

$$\therefore \quad \frac{g(y) - g(y_0)}{y - y_0} = \frac{x-c}{y-y_0} = \frac{x-c}{f(x) - f(c)}$$

$$= \frac{1}{f'(c) + \alpha(x)} \quad \cdots (II)$$

$\because$ f is diff. at c $\therefore$ f is continuous at c

$\Rightarrow$ f^{-1} is continuous at $f(c)$

$\Rightarrow$ g is continuous at $f(c)$

$\Rightarrow$ $g(y) \to g(y_0)$ as $y \to y_0$

i.e $\quad x \to c \quad$ as $y \to y_0$

so that $\alpha(x) \to 0$ as $y \to y_0$

Taking limits as $y \to y_0$ on both sides of (1)

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{f'(c) + \alpha(x)}$$

$$\Rightarrow \quad \cancel{g'(y_0)} = \frac{1}{f'(c)}$$

$$\Rightarrow \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \text{ exists}$$

Hence $g = f^{-1}$ is diff at $y = y_0$ i.e $f(a)$ and

$$g'(y_0) = \frac{1}{f'(c)} \quad \text{or} \quad g'(f(a)) = \frac{1}{f'(a)}.$$

Theorem 4.4 [Caratheodory's Theorem] Let $f : I \to R$, where I be an interval be a function on I ~~containing~~ containing the point c. Then f is differentiable at c iff there exists a function h on I that is continuous and satisfies $f(x) - f(c) = h(x)(x - c)$, for $x \in I$

Proof: If $f'(c)$ exists, we can define h by

$$h(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{for } x \neq c \\ f'(c) & \text{for } x = c \end{cases}$$

~~Since h is continuous at c~~

~~therefore, $\lim_{x \to c} h(x) = f'(c)$~~

~~If $x = c$, then LHS of~~

Now $\lim_{x \to c} h(x) = \lim_{x \to c} \dfrac{f(x) - f(c)}{\cancel{f}(x - c)}$

$$= f'(c) \quad [\because f \text{ is diff. at } c]$$

But at $x = c$, $h(c) = f'(c)$

Hence $\lim_{x \to c} h(x) = h(c)$

$\Rightarrow$ h is continuous at c and

$$h(x) = \frac{f(x) - f(c)}{x - c}$$

$\Rightarrow$ $f(x) - f(c) = (x - c)\, h(x)$.

Conversely, Assume that a function h is continuous at c and satisfies

$$f(x) - f(c) = h(x)(x - c) \qquad -\text{(I)}$$

From (I) $\dfrac{f(x) - f(c)}{x - c} = h(x)$

$$\Rightarrow \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} h(x) \qquad -\text{(II)}$$

Since h is continuous at c, therefore

$$\lim_{x \to c} h(x) = h(c) \qquad -\text{(III)}$$

From (II) and III, we get

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = h(c) \quad \text{exists}$$

$\Rightarrow$ f is differentiable at c.

Theorem 4.5 (Chain Rule) Let $f : I \to R$ and $g : J \to R$, where I and J ~~are~~ be intervals in R be functions such that $f(I) \subseteq J$. If f is differentiable at $c \in I$ and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Proof: Given that f is differentiable at $c \in I$,
So by Caratheodory's Theorem, $\exists$ a function h on I such that h is continuous at c and

$$f(x) - f(c) = h(x)(x-c), \text{ for } x \in I \quad \text{--- (I)}$$

where $h(c) = f'(c)$.
Also given that g is differentiable at $f(c)$

$\Rightarrow g'(f(c))$ ~~exist~~ exists

By Caratheodory's theorem, $\exists$ a function ϕ on J such that ϕ is continuous at $f(c)$ and

$$g(y) - g(f(c)) = \phi(y)(y - f(c)), y \in J \quad \text{...(II)}$$

where $\phi(f(c)) = g'(f(c))$

Put $y = f(x)$ in II.

$$g(f(x)) - g(f(c)) = \phi(f(x))(f(x) - f(c))$$

$$\Rightarrow (g \circ f)(x) - (g \circ f)(c) = (\phi \circ f)(x) \cdot h(x)(x-c)$$

(from I)

for all $x \in I$ s.t. $f(x) \in J$

Since $(\phi \circ f) \cdot h$ is continuous at c and

$$(\phi \circ f)(c)\, h(c) = \phi(f(c))\, h(c)$$
$$= g'(f(c))\, f'(c) \quad \text{(from I, II)}$$

By Caratheodory's Theorem, $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = (\phi \circ f)(c)\, h(c)$$
$$= g'(f(c))\, f'(c).$$

Theorem 4.6 [Darboux's Theorem] If f is differentiable on $[a, b]$ and k is a number between $f'(a)$ and $f'(b)$, then $\exists$ at least one point $c \in (a, b)$ such that $f'(c) = k$

Proof: Let $g(x) = f(x) - kx$, $x \in [a, b]$

Now g is differentiable ~~at~~ on $[a, b]$ as f & kx be differentiable on $[a, b]$ and

$$g'(a) = f'(a) - k,$$
$$g'(b) = f'(b) - k$$

Since k lies between $f'(a)$ and $f'(b)$

$\therefore$ $g'(a)$ and $g'(b)$ are of opposite sign

i.e. $g'(a) \cdot g'(b) < 0$

Therefore, $\exists$ at least one point $c \in (a, b)$ such that $f'(c) = k$.

Example 4.1 Prove that the function $f(x) = |x| + |x-1|$ is not differentiable at $x=0$ and $x=1$ but continuous at $x=0$ and $x=1$

Solution

$$f(x) = |x| + |x-1| = \begin{cases} -x-(x-1) = 1-2x, & \text{if } x<0 \\ x-(x-1) = 1, & \text{if } 0 \le x < 1 \\ x+(x-1) = 2x-1, & \text{if } x \geq 1 \end{cases}$$

At $x=0$

$$LHL = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} 1-2(-h) = 1$$

$$RHL = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (1) = 1$$

$\Rightarrow \lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} f(x) = 1$

But $f(0) = 1$

$\therefore \lim_{x\to 0} f(x) = f(0) \Rightarrow f$ is continuous at $x=0$

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{1-2(0-h) - 1}{-h}$$

$$= \lim_{h\to 0} \frac{2h}{-h} = -2$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$$

$\Rightarrow Lf'(0) \neq Rf'(0)$

$\Rightarrow f(x)$ is not differentiable at $x=0$

At $x=1$

$$LHL = f(1-0) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} (1) = 1$$

$$RHL = f(1+0) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2(1+h)-1 = 1$$

$\Rightarrow$ $LHL = RHL = 1$

$\Rightarrow$ $\lim_{x\to 1} f(x)$ exist and $\lim_{x\to 1} f(x) = 1$

But $f(1) = 1$

Hence $\lim_{x\to 1} f(x) = f(1)$

$\therefore$ $f(x)$ is continuous at $x=1$

$$L f'(1) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{1-1}{-h} = 0$$

$$R f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[2(1+h)-1] - 1}{h} = \lim_{h\to 0} 2 = 2$$

$\Rightarrow$ $L f'(1) \neq R' f'(1)$

Hence $f(x)$ is not differentiable at $x=1$.