1.1 ~~SETS~~ Sets

A well defined collection of objects is said to be set. In other words, an aggregate or class of objects having specified property in common which enables us to tell whether any given object belongs to it or not. The individual objects of the set are called members or elements of the set. Capital letters A, B, C, etc., are generally used to denote the sets while small letters a, b, c etc; for elements of sets. If x is an element of a set A, then we write $x \in A$ and read it as x belongs to A. If x is not an element of a set A, then we write $x \notin A$ and read it as x does not belong to A.

Several special sets are used throughout this book, and they are denoted by standard symbols.

- The set of natural numbers $N := \{1, 2, 3, \ldots\}$
- The set of integers $Z := \{0, 1, -1, 2, -2, 3, -3, \ldots\}$
- The set of rational numbers $Q := \{\frac{m}{n} \mid m, n \in Z \text{ and } n \neq 0\}$
- The set of real numbers R

Null Set

A set having no element is called the null set or void set or empty set. The null set is generally ~~do~~ denoted by the Danish letter ϕ or $\{\}$

For example,

I. $\phi = \{n \mid n \text{ is natural number less than } 1\}$

II. $\phi = \{x \mid x \neq x\}$

Equal Sets

Two sets A and B are called equal if they have exactly the same elements and we write $A = B$

For example,

I. If $A = \{1, 3, 7\}$ and $B = \{7, 1, 3\}$
then $A = B$.

II. $A = \{x \mid x \text{ is +ve integer and } x^2 < 40\}$
$B = \{x \mid x \in N \text{ and } x < 7\}$
then $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 2, 3, 4, 5, 6\}$
$\therefore A = B$.

Finite and Infinite Set

A set is said to be finite set if the number of elements in it is finite and a set is said to be infinite set if the number of elements in it is infinite.

For example

I The set $A = \{1, 2, 7, 9, 11\}$ is finite

II The sets N, Z, R are infinite.

Notation

$$\forall,\ \exists,\ \Rightarrow,\ \Leftrightarrow,\ \wedge,\ \vee$$

(I) $\forall$ stands for 'for all' or 'for every'

(II) $\exists$ stands for 'there exists'

(III) $\Longrightarrow$ stands for 'implies that'

IV $\Longleftrightarrow$ stands for " if and only if' or 'iff'

V $\wedge$ " " 'and'

VI $\vee$ " " 'or'

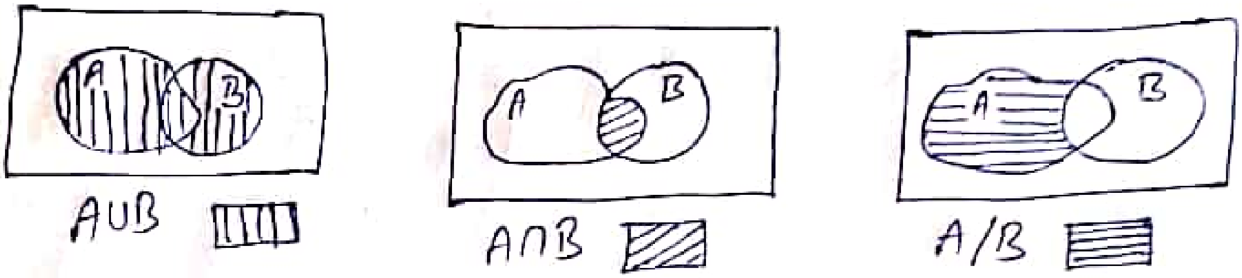
Union and Intersection of sets

(a) The union of sets A and B, denoted by $A \cup B$, is defined as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

(b) The intersection of set A and B, denoted by $A \cap B$, is defined as $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

(c) The complement of B relative to A, denoted by $A \setminus B$, is defined as

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$



Two sets A and B are said to be disjoint if $A \cap B = \phi$

Subsets

A set A is said to be subset of set B if each element of A is also an element of B i.e. $x \in A \Rightarrow x \in B$ and we write $A \subseteq B$

If each element of A is an element of B and $\exists$ an element $y \in B$ s.t. $y \notin A$, then A is said to be proper subset of B. and we write $A \subset B$.

Note (1) The null set ϕ is a subset of every set A and $A \subseteq A$ for every set A.

(2) ϕ & A are known as improper subset of set A.

1.2 Intervals

A subset ~~of~~ A of R (set of real numbers) is called an interval if A contains (I) at least two distinct elements and (II) every element lies between any two members of A

Open interval

If $a, b \in R$ satisfy $a < b$, then ~~the set~~ open interval denoted by ~~$\{x \in R \mid a \le x < b\}$~~ (a, b), is defined as

$$(a, b) = \{x \in R \mid a < x < b\}$$

The point a and b are called the end points of the interval; however, the endpoints are not the members of open interval.

Closed interval

If $a, b \in R$ s.t. $a < b$, then closed interval, denoted by $[a, b]$, is defined as the set

$$[a, b] = \{x \in R \mid a \le x \le b\}$$

The endpoints a and b are the members of closed interval.

Semi Closed or Semi open interval

The intervals $(a, b] = \{x \mid a < x \le b\}$

$[a, b) = \{x \mid a \le x < b\}$

are called semi closed or semi open.

1.3 Bounded and Unbounded Sets

~~A subset S of R is said~~

A set S is said to be bounded above if $\exists$ a real number M s.t.

$$x \le M, \ \forall x \in S$$

The number M is known as an upper bound of S.

A set S is said to be bounded below if $\exists$ a real number m s.t.

$$m \le x, \ \forall x \in S$$

The number m is known as a lower bound of S.

A set S is said to be bounded if it is bounded above as well as bounded below i.e. $\exists$ the real numbers m & M s.t.

$$m \le x \le M, \ \forall \ x \in S.$$

A set S is said to be unbounded if it is not bounded.

For examples:

(1) ~~(a)~~ Are The sets $A = \{1, 3, 5, 7, 11\}$ and $B = \{n \in N : n \le 20\}$ ~~are~~ bounded sets

Sol. $\because$ lower bound of $A = 1$

and Upper bound of $A = 11$ i.e. $1 \le x \le 11, \forall x \in A$

$\therefore$ A is bounded set

$\because$ $B = \{n \in N : n \le 20\} = \{1, 2, 3, \ldots 20\}$

lower bound of $B = 1$

Upper bound of $B = 20$, i.e $1 \le x \le 20, \forall x \in B$

$\therefore$ B is bounded set.

(2). Are the set of natural numbers and set of all integers bounded

Solution: N = The set of natural numbers

$= \{1, 2, 3, 4, \ldots\ldots\ldots\}$

$\because$ lower bound of $N = 1$

Upper bound of N = not exist

$\therefore$ N is an unbounded set.

Z = The set of all integers

$= \{\ldots\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$

The lower bound and upper bound of Z are not exist

$\therefore$ $\mathbb{Z}$ is an unbounded set.

Note: If a set S is bounded above i.e upper bound M exist, then S has an infinite number of upper bounds. Every number greater than M is also an upper bound of S. If a set S is bounded below i.e. lower bound m exist, then S has an infinite number of lower bound. Every number less than m is also ~~an~~ a lower bound of S.

Supremum of Set

(1) If a set S is bounded above, then a number M is said to be a supremum (or a least upper bound) of S, if

(i) M is an upper bound of S and

(II) $M \leq K$, $\forall$ upper bound K of S   OR

(2) If a set S is bounded above, then supremum of S, denoted by sup(S) is defined as

sup(S) = The lowest element of set of all upper bounds of S.

Infimum of Set

(1) If a set S is bounded below, then a number m is said to be an infimum (or greatest lower bound) of S if

(I) m is a lower bound of S

II $k \leq m$, $\forall$ lower bound k of S   OR

(2) If a set S is bounded below, then infimum of S, denoted by inf(S), is defined as

inf(S) = The greatest element of set of lower bounds of S

4. COMPLETENESS IN THE SET OF REAL NUMBERS

(8)

We have seen that all the properties—the properties of an ordered field, described so far, are possessed by the two sets, the set of real numbers **R** and the set of rational numbers **Q**. We shall now state a property, the property of *completeness* (or *order-completeness*) which is possessed by **R** and not by **Q**. This property not only distinguishes **R** from **Q**, but together with the ordered field property, it characterises **R** *i.e.*, the set of real numbers is the only set which is a *Complete Ordered Field.*

4.1 Order-Completeness in R

(O-C) *Every non-empty set of real numbers which is bounded above has the supremum (or the least upper bound) in* **R**.

In other words, the set of upper bounds of a non-empty set of real numbers bounded above has the smallest member.

If S is a set of real numbers which is bounded above, then by considering the set $T = \{x : -x \in S\}$ we may state the completeness property in the alternative form as:

Every non-empty set of real numbers which is bounded below has the infimum (or *g.l.b.*) *in* **R**. Or, equivalently the set of lower bounds of a non-empty set of real numbers bounded below has the greatest member.

We have thus completed the description of the set of real numbers as a **Complete Ordered Field.** We shall, however, show that the property of *completeness* does not hold good for the ordered field of rational numbers, *i.e.*, the ordered field **Q** of rationals is not order complete.

Theorem 1. *The set of rational numbers is not order-complete.*

To show that the set of rational numbers does not possess the property of completeness, it will suffice to show that there exists a non-empty set S of rationals (a subset of **Q**) which is bounded above but does not have a supremum in **Q**, *i.e.*, no rational number exists which can be the supremum of S.

Let S be the set (a subset of **Q**) of all those positive rational numbers whose square is less than 2, *i.e.*,

$$S = \{x : x \in \mathbf{Q}, x > 0 \wedge x^2 < 2\}$$

Since $1 \in S$, the set S is non-empty.

Clearly 2 is an upper bound of S, therefore, S is bounded above.

Thus, S is a non-empty set of rational numbers, bounded above. Let, if possible, the rational number K be its least upper bound. Clearly K is positive. Also by the law of trichotomy (0–1) which holds good in **Q**, one and only one of (*i*) $K^2 < 2$, (*ii*) $K^2 = 2$, (*iii*) $K^2 > 2$ holds.

(*i*) $K^2 < 2$. Let us consider the positive rational number

$$y = \frac{4+3K}{3+2K}$$

Then

$$K - y = K - \frac{4+3K}{3+2K} = \frac{2(K^2-2)}{3+2K} < 0$$

$$\Rightarrow y > K$$

Also, (1)

$$2 - y^2 = 2 - \left(\frac{4+3K}{3+2K}\right)^2 = \frac{2-K^2}{(3+2K)^2} > 0$$

$$\Rightarrow y^2 < 2 \Rightarrow y \in S$$

Thus, the member y of S is greater than K, so that K cannot be an upper bound of S and hence, there is a contradiction. (2)

(*ii*) $K^2 = 2$. We have already shown that there exists no rational number whose square is equal to 2. Thus, this case is not possible.

(10)

(*iii*) $K^2 > 2$. Considering the positive rational number y as defined in case (*i*), we may easily deduce from (1) and (2) respectively that

$$y < K \text{ and } y^2 > 2$$

Hence, there exists an upper bound y of S smaller than the least upper bound K, which is a contradiction.

Thus, none of the three possible cases holds. Hence, our supposition that a rational number K is the least upper bound of S is wrong. Thus, no rational number exists which can be the least upper bound of S.

Note. If we admit K in **R** and regard S as a set of real numbers then by the order completeness property, the supremum K of S exists in **R**. Clearly $K > 0$ and

$$K^2 < 2 \Rightarrow y^2 < 2 \wedge y > K \Rightarrow K \neq \text{Sup } S$$

$$K^2 > 2 \Rightarrow y^2 > 2 \wedge y < K \Rightarrow K \neq \text{Sup } S$$

Thus by property 0–1, it follows that $K^2 = 2$, *i.e.*, the least upper bound K exists whose square is equal to 2. Further, since $K \notin \mathbf{Q}$, it follows that K is an irrational number. Similarly, it may be seen that there exist real numbers other than rational numbers whose squares are 2, 5, 7, ... etc. This establishes the *existence of irrational numbers.*

Ex. Show that the set of natural numbers is order-complete.

4.2 Archimedean Property of Real Numbers

The order-completeness property has important consequences, one of which is the Archimedean property of real numbers which we now proceed to prove.

Theorem 2. *The real number field is Archimedean, i.e., if a and b be any two positive real numbers then there exists a positive integer n such that na > b.*

Let a, b be any two positive real numbers. Let us suppose, if possible, that for all positive integers $n (\in \mathbf{I}^+)$, $na \leq b$.

Thus, the set $S = \{na : n \in \mathbf{I}^+\}$ is bounded above, b being an upper bound. By the completeness property of the ordered-field of real numbers, set S must have the supremum M.

$$\therefore \quad na \leq M, \ \forall\, n \in \mathbf{I}^+$$

$$\Rightarrow (n+1)\, a \leq M, \ \forall\, n \in \mathbf{I}^+$$

$$\Rightarrow na \leq M - a, \ \forall\, n \in \mathbf{I}^+$$

i.e., $M - a$ is an upper bound of S.

Thus a number, $M - a$ less than the supremum M (l.u.b.) is an upper bound of S, which is a contradiction and hence our supposition is wrong.

Hence the theorem.

Corollary 1. If a be a positive real number and b any real number then there exists a positive integer n such that $na > b$.

Corollary 2. For any positive real number a there exists a positive integer n such that $n > a$.

The result follows by considering the two positive real numbers 1 and a.

Corollary 3. For any $\varepsilon > 0$ there exists a positive integer n, such that $1/n < \varepsilon$.