3.1 Continuous functions

Definition 1. A function f defined on an interval $[a,b]$ is said to be continuous at a point c, $c \in (a,b)$ if for each $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - f(c)| < \epsilon, \text{ when } |x-c| < \delta$$

In other words, A function f is said to be continuous at a point c, $c \in (a,b)$ if

(I) $f(c)$ is defined

(II) $\lim_{x \to c} f(x)$ exist i.e $f(c-) = f(c+)$

III $\lim_{x \to c} f(x) = f(c)$. i.e limit equals the Value of the function at $x = c$.

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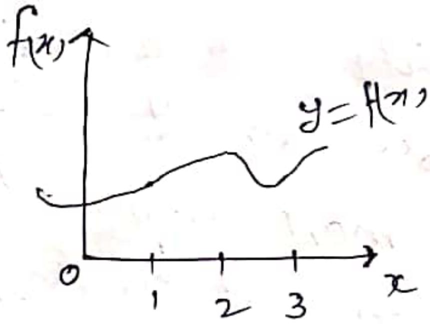
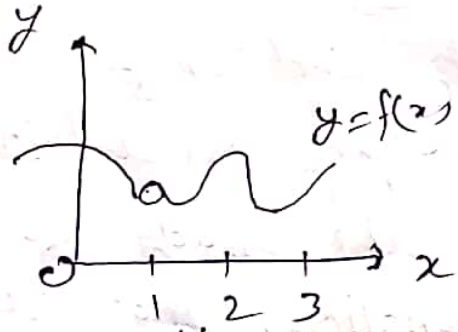
Graphically, a function $f(x)$ is continuous at $x=c$ if its graph has no break at $x=c$



(a)
continuous at $x=1$



(b)
not continuous at $x=1$

Note 1. (a) function is continuous at $x=1$, because its graph has no break at $x=1$

(2) (b) The function is not continuous at $x=1$, because its graph has break at $x=1$

Definition 2. A function f defined on $[a,b]$ is said to be continuous from the left and from the right at a point c if

$$\lim_{x\to c-0} f(x) = f(c) \ \& \ \lim_{x\to c+0} f(x) = f(c) \text{ respectively}$$

Clearly a function defined on $[a,b]$ is continuous at a point c iff it is continuous from left as well as ~~the~~ from right.

Definition 3: A ~~continuous~~ function f defined on [a,b] is said to be continuous at the end point a if

$$\lim_{x \to a+0} f(x) = f(a)$$

Also f is continuous at end point b if

$$\lim_{x \to b-} f(x) = f(b).$$

Definition 4: A function f defined on [a,b] is said to be continuous in an interval [a,b] if it is continuous at every point of the interval.

3.2 Discontinuous functions

A function f defined on [a,b] is said to be discontinuous at a point c if it is not continuous at a point c and a point c is called a point of discontinuity of the function.

Type of discontinuities

(1) Removable discontinuity

A function f defined on [a,b] is said to have a removable discontinuity at a point c if $\lim_{x \to c} f(x)$ ~~exists~~ exists (i.e. $f(c-) = f(c+)$) but is not equal to $f(c)$ (which may or may not be exist).

(2) Discontinuity of the first kind [or Jump discontinuity]

A function f is said to have discontinuity of first kind at a point c if LHL and RHL at a point c both exist but ~~HLH~~ LHL at c $\neq$ RHL at c

i.e $f(c-0) \neq f(c+0)$

(3) Discontinuity of the second kind

A function f is said to have discontinuity of second kind at a point c if neither LHL at c nor RHL at c exists.

Example 3.1 ~~A function f defined on R is given by~~ $f(x) = |x|$, for $x \neq 0$ and $f(x) = 0$ for $x=0$ ~~f(x) = |x| for Examine f~~ ~~for continuity~~ at $x=0$

~~Solution:~~

Example 3.1 If a function $f(x) = |x|$, for $x \neq 0$ and $f(x) = 0$ for $x=0$. Examine f for continuity at $x=0$.

Solution: The given function can be written as

$$f(x) = \begin{cases} -x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ x & \text{if } x>0 \end{cases}$$

~~LHL at x=0~~

$$LHL = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} -(0-h)$$

$$= \lim_{h\to 0} h = 0$$

~~RHL at x=0~~

$$RHL = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (0+h)$$

$$f(0+0) = \lim_{h\to 0}(h) = 0$$

Thus $f(0-0) = f(0+0) = 0$

$\Rightarrow \lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} f(x) = 0$ — (I)

Given $f(0) = 0$ — (II)

From (I) and II, we get $\lim_{x\to 0} f(x) = f(0)$

Hence the given function f is continuous at $x = 0$.

Example 3.2 Show that the function defined on R by

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is continuous at $x=0$

Solution:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x \sin\frac{1}{x}\right)$$

$$= \lim_{x\to 0}(x) \cdot \lim_{x\to 0}\left(\sin\frac{1}{x}\right)$$

$$= 0 \times [\text{a finite value oscillating between } -1 \,\&\, 1]$$

$$= 0$$

$\Rightarrow \lim_{x\to 0} f(x) = 0$ — (I)

Given that $f(0) = 0$ — II

From (I) & II, $\lim_{x\to 0} f(x) = f(0)$

Hence given function $f(x)$ is continuous at $x=0$.

Example 3.3 Examine the continuity at $x=0$

$$f(x) = \begin{cases} (1+x)^{\frac{1}{x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Solution: $\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

$$= \cancel{\lim_{x \to 0}} \; e^{\lim_{x \to 0} \left(x \times \frac{1}{x}\right)}$$

$$= e^{\lim_{x \to 0}(1)} = e$$

$\Rightarrow \lim_{x \to 0} f(x) = e$ —— (I)

Given that $f(0) = 0$ —— II

From (I) & II, $\lim_{x \to 0} f(x) \neq f(0)$

Hence the given function $f(x)$ have removable discontinuity at $x=0$

Example 3.4 Examine the discontinuity at $x=1$ for the function, $f(x) = [x]$, $\forall\ x \geq 0$, where $[x]$ denotes the largest integer $\leq x$.

Solution
LHL $= f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} [1-h]$

$$= \lim_{h \to 0} 0 = 0$$

RHL $= f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [1+h]$

$\Rightarrow f(1+0) = \lim_{h\to 0} 1 = 1$

Thus LHL at $x=1$ and RHL at $x=1$ both are exist but $f(1-0) \neq f(1+0)$

Hence the given function f have discontinuity of first kind at $x=1$

Example 3.5 Prove that the Dirichlet's function f defined on R by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

Solution: Let a is any rational number, $f(a) = 1$

Since there lie an infinite number of rational and irrational number in any interval, therefore we can choose a sequence $\{x_n\}$ of irrational number that converges to a

Since $f(x_n) = 0, \quad \forall n \in N$

therefore $\lim_{n\to\infty} f(x_n) = 0$

~~But f~~

$\Rightarrow$ ~~$|f(x_n) - 0| < \epsilon$, when $|x_n - a|$~~

But $f(a) = 1$

Therefore, f is discontinuous at a

$\Rightarrow$ f is discontinuous at all rational numbers.

Next, let b is any irrational numbers, then

$f(b) = 0$,

We can choose a sequence $\{x_n\}$ of rational numbers that converges to b.

Since $f(x_n) = 1 \;\; \forall\, n \in N$

Therefore, $\lim f(x_n) = 1$ while $f(b) = 0$

Hence f is discontinuous at b

$\Rightarrow$ f is discontinuous at all irrational numbers

Therefore f is discontinuous at every point.

Example 3.6 Discuss the continuity at $x=-2$, and $x=2$ of the function f defined by

$$f(x) = \begin{cases} -x^2 & , \text{ if } x \le -2 \\ 4 & , \text{ if } -2 < x \le 2 \\ x^2 & , \text{ if } x > 2 \end{cases}$$

Solution At $x=-2$, $f(-2) = -(-2)^2 = -4$

$$LHL = f(-2-0) = \lim_{h\to 0} f(-2-h) = \lim_{h\to 0} -(-2-h)^2$$

$$= -(-2-0)^2 = -4$$

$$RHL = f(-2+0) = \lim_{h\to 0} f(-2+h) = \lim_{h\to 0}(4) = 4$$

$\Rightarrow LHL \neq RHL$ at $x=-2 \Rightarrow \lim_{x\to -2} f(x)$ does not exist

Hence, $f(x)$ is not continuous at $x=-2$

At $x=2$

$f(2) = 4$

$$LHL = f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}(4) = 4$$

$$RHL = f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} (2+h)^2$$

$$= \lim_{h\to 0} (2+0)^2 = 4$$

$$\Rightarrow f(2-0) = f(2+0) = 4$$

$$\Rightarrow \lim_{x\to 2} f(x) = 4 = f(2)$$

Hence $f(x)$ is continuous at $x=2$

Example 3.7 Investigate the continuity at $x=0$ of the function f, where

$$f(x) = \begin{cases} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} & , \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0 \end{cases}$$

Solution $LHL = f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$

$$= \lim_{h\to 0} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} = \lim_{h\to 0} \frac{e^{1/h}(e^{-2/h}-1)}{e^{1/h}(e^{-2/h}+1)}$$

$$= \lim_{h\to 0} \frac{e^{-2/h}-1}{e^{-2/h}+1} = \frac{0-1}{0+1} = -1$$

$RHL = f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

$$= \lim_{h\to 0} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = \lim_{h\to 0} \frac{e^{1/h}(1-e^{-2/h})}{e^{1/h}(1+e^{-2/h})}$$

$$= \lim_{h\to 0} \frac{1-e^{-2/h}}{1+e^{-2/h}} = \frac{1-0}{1+0} = 1$$

$\Rightarrow f(0-0) \neq f(0+0)$

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist

Hence $f(x)$ is not continuous at $x=0$

2. Show that $f(x)$,

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 0 \\ 1, & \text{if } \cancel{x=0}\ 0 < x \leq 1 \\ 1/x, & \text{if } x > 1 \end{cases}$$

is ~~for~~ discontinuous at $x=0$ and continuous at $x=1$

3. Show that the function f defined on R, where

$$f(x) = \begin{cases} -1, & \text{if } x \in Q \\ +1, & \text{if } x \in Q^c \end{cases}$$

Q is set of rational numbers, is discontinuous at every point

4. Examine the function defined below for continuity at $x=a$

$$f(x) = \frac{1}{x-a} \operatorname{cosec}\left(\frac{1}{x-a}\right), \quad x \neq a \text{ and } f(x) = 0, \ x = a$$

5. Discuss the continuity at $x=0$ of the function f, where $f(x) = \frac{|x|}{x}$ for $x \neq 0$ and $f(0) = 0$

6. A function f is defined as follows

$$f(x) = \begin{cases} (x^2/a) - a, & \text{if } x < a \\ 0, & \text{if } x = 0 \\ a - (x^2/a), & \text{if } x > a \end{cases}$$

Show that the function is continuous at $x=a$