Supremum of Set

1

(1) If a set S is bounded above, then a number M is said to be a supremum (or a least upper bound) of S if

(i) M is an upper bound of S and

(II) $M \leq K$, $\forall$ upper bound K of S OR

(2) If a set S is bounded above, then supremum of S, denoted by $\sup(S)$ is defined as

$\sup(S)$ = The lowest element of set of all upper bounds of S.

Infimum of Set

(1) If a set S is bounded below, then a number m is said to be an infimum (or greatest lower bound) of S if

(I) m is a lower bound of S

II $k \leq m$, $\forall$ lower bound k of S OR

(2) If a set S is bounded below, then infimum of S, denoted by $\inf(S)$, is defined as

$\inf(S)$ = The greatest element of set of lower bounds of S

Note (1) sup(S) may or may not exist and in case it exists, it may or may not belong to S.

II sup(S) is unique i.e. a set can not have more than one supremum.

III inf(S) may or may not exist and in case it exists, it may or may not belong to S.

IV inf(S) is unique i.e a set can not have more than one infimum.

For examples

(1) Find the infimum and the supremum of the following sets. Which of these belongs to the set ?

(I) $\{0, 2, 4, 6, 8\}$ (II) $\{\frac{1}{n} \mid n \in N\}$

(III) $\{\frac{(-1)^n}{n} \mid n \in N\}$ (IV) $\{-\frac{n+1}{n} \mid n \in N\}$

Solution

(I) Given $A = \{0, 2, 4, 6, 8\}$

$\inf(A) = 0$ as 0 is greatest element of set of lower bounds of A.

$\sup(A) = 8$

Clearly inf(A) and sup(A) both are belong to A

(II) Given $A = \{\frac{1}{n} \mid n \in N\}$

$= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$\inf(A) = \frac{1}{n}$ as $n \to \infty$

$= 0 \notin A$

$\Rightarrow \inf(A) \notin A$

$\sup(A) = 1 \in A$

$\Rightarrow \sup(A) \in A.$

III. Given $A = \left\{ \frac{(-1)^n}{n} \mid n \in N \right\}$

$$= \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\ldots \right\}$$

$$= \left\{ -1, -\frac{1}{3}, -\frac{1}{5}, \ldots\ldots, \frac{1}{6}, \frac{1}{4}, \frac{1}{2} \right\}$$

$\inf(A) = -1$

$\sup(A) = \frac{1}{2}$

Hence $\inf(A) \in A$, $\sup(A) \in A$.

IV. Given $A = \left\{ -\frac{n+1}{n} \mid n \in N \right\}$

$$= \left\{ -2, -\frac{3}{2}, -\frac{4}{3}, \ldots\ldots \right\}$$

$\inf(A) = -2$, $\sup(A) = -\frac{n+1}{n}$ as $n \to \infty$
$= -1$

Hence $\inf(A) \in A$ and $\sup(A) \notin A$.

1.4 Neighbourhood of a point

A set $S \subseteq R$ is called a neighbourhood of a point x, if $\exists$ an open interval I s.t.

$$x \in I \subseteq S$$

and it is denoted by $nbd(x)$

It follows from the definition that neighbourhood of a point x, ~~denoted by~~ is an open interval

i.e. $nbd(x) = (x-\delta, x+\delta)$, where $\delta > 0$

For examples

(I) The set R of real numbers is the neighbourhood of each ~~points~~ of its points.

(II) The set N of natural numbers is not the nbd. of each of its points.

(III) The open interval (a, b) is nbd. of each of its
IV The closed interval $[a, b]$ is nbd of each of its points except the endpoints a and b.
V A non empty finite set is not a nbd of any point.

1.5 Limit Points

Definition 1. A real number ξ is said to be a limit point of a set S if every nbd of ξ contains an infinite number of members of S.
A limit is also called a cluster point, a condensation point or an accumulation point.

Definition 2. A real number ξ is said to be a limit point of a set S. if every nbd of ξ contains at least one member of S other than ξ.

A limit point of a set may or may not be a member of set. Further it is clear ~~that~~ from the definition that a finite set can not ~~be~~ have a limit point.

Example 1. Show that the set Z of integers has no limit point.
Solution: Let $m \in Z$, then $(m-\frac{1}{2}, m+\frac{1}{2})$ is a nbd of m
Since $(m-\frac{1}{2}, m+\frac{1}{2})$ has no integers of Z other than m.
$\therefore$ m is not limit point of Z.

m is an arbitrary ~~ingers~~ integers of $\mathbb{Z}$ (11)
$\Rightarrow$ $\mathbb{Z}$ has no limit point.

Theorem (Bolzano - Weierstrass Theorem) Every infinite bounded set has a limit point.

Proof: Let S be any infinite, bounde set and m, M are its infimum and supremum respectively.
Define a set P of real numbers s.t.

$x \in P \iff$ it exceeds at the most a finite number of S.

~~The~~ The set $P \neq \phi$, for $m \in P$. Also M is an upper bound of P, for ~~not~~ no number greater than or equal to M can belong to P.
thus P is bounded above $\Rightarrow$ P has supremum, say ξ.
To prove that ξ is limit point of ~~δ~~ S.
Consider ~~and~~ any nbd. $(\xi - \epsilon, \xi + \epsilon)$ of ξ, where $\epsilon > 0$
Since ξ is the $\sup(P)$, $\exists$ at least one number say $h > \xi - \epsilon$. Now $h \in P$ $\therefore$ it exceed at the most a finit number of member of S, and $\xi - \epsilon$ can exceed at the most a finite number of members of S. Again $\xi = \sup(P)$, $\xi + \epsilon$ cannot belong to P and $\xi + \epsilon$ must exceed an infinite number of member of S.
Now $\xi - \epsilon$ exceed at the most a finite number of members of S. and $\xi + \epsilon$ exceeds infinitely many members of S.
$\Rightarrow (\xi - \epsilon, \xi + \epsilon)$ contains an infinite number of

members of S.

$\Rightarrow$ ξ is a limit point of S.

Hence every infinite bounded set has a limit point.

(1) Find the limit points of the set $\{1-\frac{1}{n} : n \in N\}$

(2) Obtain the limit points of the set

$\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, \ldots\}$

(3) Obtain the limit points of the ~~set~~ closed interval $[a,b]$ and open interval (a,b)

(4) Obtain the limit points of the set N of natural number and set Q of rational numbers.

(5) Obtain the limit point of $\{\frac{1}{n}; : n \in N\}$

Answers

(1) 1 (2) 1, -1 (3) Set of limit point of $[a,b] = [a,b]$ and $(a,b) = [a,b]$

(4) no limit point, and (5) 0
All real numbers.